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On the Role of Applied Mathematics

C. C. LIN

Massachusetts Institute of Technology, Cambridge, Massachusetts

TO MY GOOD FRIEND STANISLAW MARCIN ULAM IN HONOR OF HIS 65TH BIRTHDAY

This paper concerns itself with the nature of applied mathematics, and how to make its role effective. Its differences from pure mathematics and from theoretical sciences are pointed out with the help of illustrative examples. Recommendations are made for supporting a program of education and research in applied mathematics to make it an independent scientific discipline and to create a community of applied mathematicians.

I. INTRODUCTION

1. *The Role of Applied Mathematicians*

It is a pleasure for me to be given this opportunity to speak to this distinguished audience on the role of applied mathematics—even though I would rather, like everybody else, speak about my own research work on stellar systems, on galaxies, on hydrodynamics and on turbulence—because I do have a few thoughts to share with you regarding the profession of applied mathematics as a whole. Every applied mathematician here knows what he is doing in his own line of specialty, and it would be presumptive for me to comment on that. However, I feel that there are things which are left unattended and which should be done by all applied mathematicians as a group to enhance the role of our profession. There are also several rather philosophical issues which, I feel, deserve our continued attention, because they would influence the long-term outlook of applied mathematics. It is toward these issues that I shall address my discussion. I hope that my remarks are also of interest to the pure mathematicians and to the theoretical and empirical scientists, with whom the applied mathematicians often develop close working relationships.

I should add that, like anybody else, my opinions will be biased by my own experience, which does not include any extensive stay in a laboratory such as this one. I hope, therefore, that I shall be excused and corrected if my remarks do not quite fit the conditions here.

It is almost tautological to say that the role of the applied mathematician is to promote the most effective use of mathematics in the natural and social sciences, in applied sciences, in engineering, in industry, in government, and in all other kinds of human activity. But this is not the complete list of his professional activities. As an intelligent scientist, he can also ask intelligent questions and originate new ideas which are only remotely or partly mathematical but which are nevertheless extremely important. I have in mind such ideas and practical inventions as those pioneered by our friend Stan Ulam at this laboratory.

However, in this talk, I shall restrict myself more narrowly to the effective use of mathematics in the sciences and to the stimulation of pure mathematical research through such efforts. One of the focal points of my discussion will be *how* to make this role effective.

In the course of this discussion, I shall comment on a few philosophical issues for which there seem to be, as yet, no generally accepted opinion. This is not surprising: philosophy is a subject in which it is easy to ask good questions and difficult to find good answers. The questions I would like to ask are the following: (a) Applicability: Why is mathematics so broadly applicable?[1] (b) Universality: Are the mathematics usable in one science also most likely to be usable in another? (c) Probability of applicability: (i) Can we identify a body of knowledge known as "applicable mathematics"? (ii) Is mathematics ever to be applicable to such subjects as history?

I shall attempt to give answers to some of these questions.

To continue with this discussion, we must first agree on an answer to the question "What is applied mathematics?" Different people will probably give answers with different emphases. I hope that my answer will be broad enough to receive general acceptance and be useful as a basis for our discussion.

[1] Cf. [1].

II. What is Applied Mathematics?

2. *The Nature and Scope of Applied Mathematics*

Generally speaking, applied mathematics is a disciplined activity which lies between the empirical sciences and pure mathematics. In essence, it is characterized by an attitude, an approach, a way of thinking. The principal theme is the *interdependence* of mathematics and the sciences. In common with the pure mathematician, the applied mathematician is interested in the stimulation of the development of new mathematics (see [2]),—but with primary emphasis on those aspects *directly* or at least very strongly motivated by scientific problems. In common with the theoretical scientists, the applied mathematician seeks knowledge and understanding of scientific facts and real world phenomena through the use of mathematical methods.

A particularly challenging type of activity is to find *new* ideas for the *application* of mathematics and indeed to develop mathematical theories in those scientific subjects which have not hitherto been subjected to systematic mathematical treatment. (Social sciences and biology are the oftcited examples.) These efforts may in turn lead to the creation of new mathematical ideas and theories (by abstraction, generalization, or otherwise), which are interesting in their own right as a part of pure mathematics. The recognition of this duality is essential to the spirit of applied mathematics. By stressing this duality, there is obviously a *difference in emphasis* between applied mathematics and either pure mathematics or the empirical sciences.

The scope of applied mathematics is very broad, and can be best described by borrowing the following words from Albert Einstein [3]:

> "Its realm is accordingly defined as
> that part of the sum total of our
> knowledge which is capable of being
> expressed in mathematical terms."

These words were originally used to define physics. Taken literally, the statement clearly includes the mathematical theories of economics, biology, communication, etc., and is perhaps a more adequate description of applied mathematics.

Clearly, the activity of every branch of science includes major efforts *not* deeply mathematical. For example, the *act* of observation of the behavior of sun spots and solar winds, isolation of radio-active elements, taking of opinion polls (not the analysis), and cultivation of bacteria. Thus, applied mathematics does not *include* all the sciences. It merely *overlaps* with all the sciences. The extent of this overlap may be large (as in physics) or small (as in biology, for the moment). It is indeed the extension of this overlap that should concern the applied mathematician. At the same time, the *wisdom* of the applied mathematician, generated by his intimate knowledge of mathematical methods (or "applicable mathematics," if you wish) and his experience with a *variety* of mathematical applications, puts him in the best position to judge in which areas or problems of science the power of a particular branch of mathematics would be most effective, would have only limited effectiveness, or would have no effectiveness at all. His efforts must obviously be devoted to those aspects of science where mathematical methods can be effectively used. The practitioner must *continue* to improve upon his own mathematical knowledge[2] and new mathematics must be created when necessary. Fortunately, our experience has shown that there is indeed simplicity and order in the fundamental aspects of the sciences. All fundamental laws of physics can be stated in mathematical form on one sheet of paper. At the same time, it appears to be true that human intelligence insists on conceptual simplicity in the formulation of the fundamentals of any scientific subject. *Thus, the mathematical approach is also the basic approach.* At the other extreme, mathematical methods also can be effectively used in detailed scientific or technological calculations.

To summarize, a well-educated applied mathematician must have a clear appreciation of the above-mentioned spirit of duality and a comprehensive knowledge of the total picture of the mathematical sciences. Let us now turn to a more specific discussion of his creative activities.

3. *Creative Activities of Applied Mathematicians, Part A* [3]

Applied mathematicians are often found to be engaged in the following efforts: (a) the formulation of scientific concepts and problems in mathematical terms, (b) the solution of the resultant mathematical problems, and (c) the discussion, interpretation and the evaluation of the results of his analysis, including the making of specific predictions. But the solution of specific problems often serves merely as a focus and

[2] See, for example, Theodore von Kármán's remarks at the age of 80 in [4, Preface].
[3] Cf. [5].

an aid in reaching a deeper understanding. The final goal of these efforts of an applied mathematician lies in (d) the creation of ideas, concepts, and methods that are of basic significance and general applicability to the subject in question, including the formulation of general principles. As mentioned above, these efforts may lead to (e) the creation of new mathematical ideas and theories. It might be added that, because of the origin of such mathematical theories, they are more likely to be applicable to other branches of the sciences.[4]

The basic difference in motivation between pure and applied mathematicians is reflected in the habits and practices of their activities. Although the applied mathematician understands and appreciates the nature of a rigorous demonstration, he cannot be made inactive by these considerations. In phase (b) of his activities, his primary emphasis is always directed toward the ultimate solution, and he frequently uses heuristic scientific reasoning to achieve this end. In this way, an analysis might be made more amenable to solution, approximations introduced, or arguments made plausible. His treatment is at all times responsible and disciplined, but he is not a deductive logician interested solely in the beauty of form and the power of abstraction. He should, however, have enough background in pure mathematics to be able to distinguish between rigorous proof, clear demonstration, plausible arguments, and hopeful speculation; all of which are used in the study of applied mathematics. In particular, when he is engaged in the creation of new mathematics, the applied mathematician should follow the practices (including the degree of rigor) used in pure mathematics in the *formulation* of his results. Because of his background, heuristic reasoning leading to the final form of the theory will doubtlessly be emphasized.

In many cases, the development of a rigorous form of the mathematical method or theory cannot be accomplished, yet the method has to be used with only plausible arguments to support its reliability. Most work in numerical solution of scientific problems belongs to this category.

In the other two phases of his activity, (a) and (c), the applied mathematician should follow the practices of a theoretical scientist. *The construction of an idealized mathematical model is indeed the most important and the most difficult phase*, especially in a new field of application of

[4] Cf. Hirsh Cohen [6]. In this article, he pointed out the similarity of the nature of these mathematical problems and their solution to those in mechanics and classical applied mathematics. Also, Lawrence Klein, in [7], stated ". . . but there are few examples in which it can be said that our subject has called forth its own branch of mathematics or given inspiration for great new mathematical discoveries."

mathematics. It usually requires a comprehensive knowledge and a deep understanding of the empirical facts related to the particular phenomenon under consideration as well as penetrating insight and mature judgement. These are also required in phases (c) and (d) of his activities. In these phases, the applied mathematician should examine his results to reach a deeper understanding of the problem at hand, and attempt to abstract the essentials to form concepts which are of more general applicability. At the same time, his conclusions must, of course, be checked against the existing body of knowledge. Any new inferences or predictions from his results are also subject to verification by further experimentation and observation, since their truth cannot be determined by purely logical means. With complementary theoretical and empirical efforts, a deeper and more penetrating understanding may be achieved.

Despite these similarities of activity, there are *subtle differences in attitude between a theoretical scientist and an applied mathematician.* The theoretical physicist, for example, has his primary interest in the discovery of new physical laws. Indeed, a correct theory can be formulated only after a great deal of trial and error. (Consider the current efforts in high-energy physics.) In contrast, the applied mathematician places more emphasis on the use of mathematical methods for the description of physical phenomena in terms of known physical laws. (Consider the current efforts in numerical weather forecasting.) He is also greatly interested in the stimulation of new mathematical ideas. One might say that the difference lies in the relative extent to which *inductive* and *deductive* reasoning are emphasized in each discipline.

However, these subtle differences in motivation can lead to substantial differences in the choice of the educational program and in the attitude developed in a person, which in turn color his activities. For example, classical mechanics (including particle mechanics and continuum mechanics) is seldom studied in detail by modern students of physics, yet it remains one of the principal subjects for the education of applied mathematicians. The fundamental concepts and the mathematical methods which have been developed through such studies still constitute the essential basis for the application of mathematics, whether the particular subject in question is atmospheric turbulence, radio astronomy, or demography. Indeed, the applied mathematician does not draw sharp lines along traditional boundaries of subject matter, and it is desirable for him to adapt his interests to the present and future vitality of the subject matter if he expects his research efforts to have an impact beyond the development of applicable mathematical methods (cf. Appendix).

The desire and ability to cut across traditional scientific disciplines, through the medium of mathematics, are perhaps the unique characteristics of an applied mathematician. Indeed, the education of an applied mathematician must provide him with a breadth of knowledge in both mathematics and the fundamentals of the sciences. He is then in the best position to advance a specific subject by the creation of suitable mathematical models, by his critical and precise thinking, and by transferring the mathematical knowledge gained through the study of another scientific discipline.

The above description of an applied mathematician almost compels him to stay at the general fundamental level of a braod spectrum of sciences. Of course, an applied mathematician could, if he so chooses, become a specialist in a given scientific subject. In doing this, he should perhaps subtly change his attitude and adopt entirely the spirit of the specialists in that field.

In order to educate applied mathematicians, a program of study must be provided both in the undergraduate and the graduate levels. Opportunities for post-doctoral research should also be available. In particular, there must be a course dealing with the fundamentals of the five-step processes described at the beginning of this section, with special emphasis on the first step, the construction of suitable mathematical models in a variety of problems. This last part is usually lacking in most of the curricula in mathematics.

4. *Creative Activities of Applied Mathematicians, Part B*

Applied mathematicians are sometimes also found in the resolution of mathematical issues that arise from their scientific problems. This interest they share with the willing pure mathematicians. Although this is a less frequent type of activity than that described above, it would be unfortunate to ignore this aspect of interaction between mathematics and the sciences. To quote from Chapter II of the COSRIMS Report [8], "Fourier considered mathematics as a tool for describing nature. But the impact of Fourier, important as it was in physics and engineering, was particularly felt in some of the purest branches of mathematics."

It is also to be recalled that when Fourier submitted his mathematical theory of heat to the French Academy for the Grand Prize, the referees, Laplace, Lagrange, and Legendre pointed out that the mathematical treatment left much to be desired in the way of rigor, even though they admitted the novelty and importance of Fourier's work. More than a century was to elapse before all the subtle mathematical issues were satisfactorily resolved.[5] (To be sure, some of these issues were raised later.)

Here is a case from which much can be learned on the nature of the interaction. *The type of mathematical theory which is developed for the solution of certain scientific problems can often find application in other scientific problems.* For, when formulated in abstraction, i.e., in mathematical terms, they become similar or even identical. Secondly, in the early stages of the development of such work, its implication in pure mathematics is frequently unclear, and cannot be fully appreciated. The evaluation of such activities, therefore, must be made by the professionals as in any other discipline, to avoid unfair judgment.

In order to carry on with this tradition of Fourier, it is clear that the applied mathematician should be educated in the modern developments in pure mathematics. Even if his aim is less ambitious, he should be well informed in pure mathematics in order to make use of the latest accomplishments. In a university, special courses should be planned to introduce the essentials of pure mathematics to students in applied mathematics (cf. Sections 7 and 8 in Part III). As mentioned before, the applied mathematician also must be prepared to learn new mathematics throughout his career whenever it becomes necessary.

The contribution of pure mathematicians, who show interest in applications, can be extremely great in this area of interaction between science and mathematics. They can frequently contribute to the abstraction, consolidation, and generalization of special developments into a more elegant and a more powerful structure. Naturally, they would also contribute to *other mathematical developments which might appear to be useless at the beginning, but would later turn out to have important scientific applications.* However, I must disqualify myself from pursuing the discussion further along these lines. I hope that my friends in pure mathematics would expound on this theme further and furnish us with impressive examples.

From the standpoint of an applied mathematician, I would like, however, to add one comment. The theorems we need are often not the most general. We would be perfectly happy to have theorems proved

[5] See [9, 10].

under suitable restrictions. For Fourier analysis, piecewise smooth functions are usually sufficient. The extension to functions of bounded variation, Lebesque measurable functions, etc., are decidedly of lower importance. (For a modern problem in control theory, see [11]. Piecewise analytic functions are found to be most suitable.)

5. *Examples*

We shall now examine a few examples in physical mathematics to see how the above idealized description of the creative activities of applied mathematics fits realistic developments in specific instances.

(a) *Fourier Analysis, a Case of Unqualified Success*

First, consider the case of Fourier analysis. This is perhaps the best example of a complete "cycle" of development. A relatively simple class of problems in mathematical physics, heat conduction, was solved according to steps (a)–(c) outlined in Section 3. The mathematical theories of Fourier series, Fourier integral, and generalized harmonic analysis are very impressive developments. The theory of Lebesque measure is easily motivated by asking a simple question: Is the Converse to Parseval's theorem a true statement? Modern radio astronony, with its multimillion dollar equipments and multimillion dollar annual budgets, depends on Fourier analysis for its very operation. The process of Fourier analysis is indeed built into the automatic online data processing equipment. The spectral analysis of the signals from distant astronomical objects is actually done by calculating the autocorrelation function and its Fourier transform (see [12, Fig. 5]).

The spatial Fourier analysis of the sky is also essential to radio astronomy. For this purpose, an array of radio telescopes is needed. A very large array, shaped in the form of the letter *Y* has been planned for completion in this country as a 10-year program, with a total budget exceeding one hundred million dollars. Figure 1 shows a linear array in Holland. It has a system of 12 telescopes spaced over a distance of 1600m. While the signal received by each of the telescopes can be analyzed in time by the method described above, the use of pairs of telescopes as interferometers enables the system to give a spatial Fourier analysis of the sky along the array. As the earth spins around its axis and revolves in its orbit, different directions in the sky may be covered. The Dutch astronomers are now in the process of doubling the length of the telescope system and the number of interferometer pairs in it.

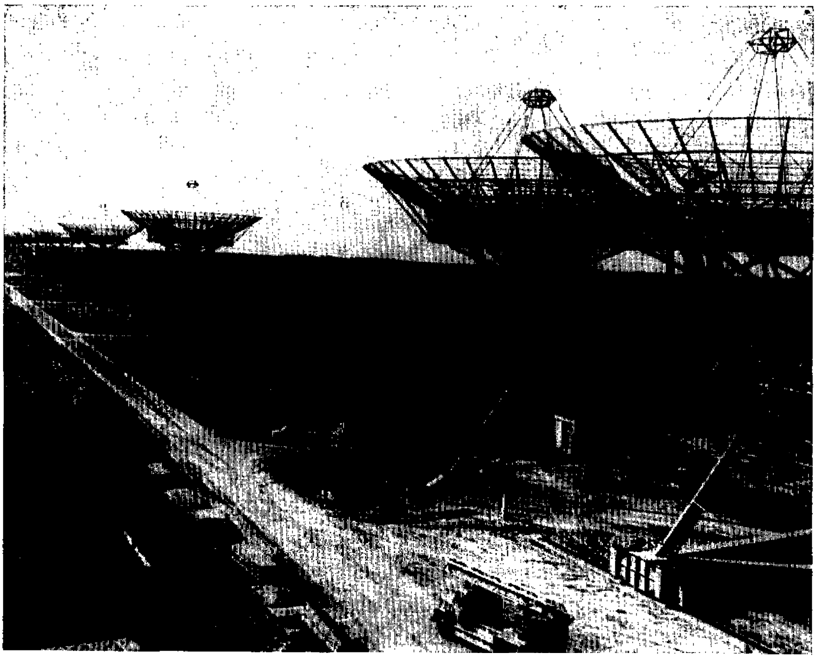

FIG. 1. Part of a row of 12 radio-telescopes, each 25 m in diameter. The system, extending over a distance of 1600 m, provides 20 interferometers for the analysis of the structure of the radio sources. Present plans call for a doubling of the distance and the number of interferometers.

At microscales, the study of molecular structure is done by the study of X-ray diffraction patterns. The analysis is again based on Fourier methods. On the theoretical side, fundamental concepts in quantum mechanics, the wave-particle duality, can be best understood by using Fourier integrals.

(b) *Turbulence, an Incomplete Picture*

However, complete success of this type is rare. In many instances, success is only partial.

Consider the problem of turbulence. Fourier analysis is a necessary part of the theory, and indeed, provides the basic language for the subject. However, there are essential difficulties elsewhere. Turbulence is basically a nonlinear random process, in which the dissipative effect plays a paramount role as the stochastic process. There is as yet no mathematical theory for turbulence despite many heroic efforts over several decades.

Even on the level of a single field of fluid motion, which is governed by a well-established set of partial differential equations (the Navier-Stokes equations) the rigorous mathematical theory provides little help. The equations are quasilinear. The existence theory for this set of partial differential equations has been established only for very small values of the Reynolds number, a characteristic dimensionless parameter of the problem. To study the theory of turbulence, one must however deal with the fluid motion at extremely large Reynolds numbers.

In the more limited problem of hydrodynamic stability, where we may linearize the equations in the first approximation, rigorous mathematical theories have been found to be more helpful. There have also been very interesting new mathematical developments stimulated by the needs in the solution of the physical problem.

(c) *Numerical Analysis, a Case of Acquiescence*

The difficulty of dealing with nonlinear partial differential equations shows up even more clearly in the need to proceed with large-scale numerical calculations, despite the lack of mathematical theorems that would guarantee the convergence of such procedure. Examples that easily come to mind include much of the work done in this laboratory on the numerical solution of fluid dynamical problems and numerical weather forecasting. Logical-minded persons have to acquiesce to such decisions to go ahead for the lack of a better solution. Applied mathematicians are of course quite at home with such approaches, because there are various ways to ensure the plausible correctness of such procedures.

(d) *Plasma Physics and Stellar Dynamics, a Case of Mutual Support Without Firm Mathematical Foundation*

Plasma physics is a subject of great interest to this laboratory: The problem of plasma containment is still an unsolved problem, Professor Harold Grad would be better qualified than I to speak on the mathematical issues. But I am also not aware of any significant contributions to this science from new mathematical theorems. Most of the theoretical progress has been made in the spirit of applied mathematics. Indeed, in some of the exploratory calculations being made now, the procedures followed would seriously worry theoretical hydrodynamicists. The substantiation of such theoretical analysis depends heavily on experimental verification. At the same time, I am informed by my M.I.T. colleagues, who are theoretical plasma physicists working closely with experimental programs, that most of the desired experiments, with very few exceptions, are yet to be done. Here is a challenging opportunity for applied mathematicians to make substantial contributions in consolidating and improving upon the methods of analysis to make the predictions more reliable.

At the same time, these theoretical calculations can derive indirect support from distant places. The study of spiral structure in galaxies has led to the investigation of collective modes in stellar systems which are essentially plasmoidal in nature. These modes have been observed in terms of their effects on star formation. To be sure, the analysis in these cases has been done quite carefully, but even modes calculated with similar care in plasma physics have not yet been observed.

I mention this case to stress the importance of reasoning by analogy in the work of applied mathematicians. Confidence and insight can be gained in our understanding of plasma physics even though the work on galaxies contains no logical deduction in the context of electromagnetic plasmas.

Indeed, the concept of collective modes might be applicable (perhaps already applied) to social sciences. Each person is analogous to a star. At least in economics, a person interacts more with the institutions than with individuals. The latter kind of interaction plays a rather minor role to influence the economics of the country as a whole.

(e) *Transonic Flow, a Case of Mismatch*

When engineers began to design airplanes to fly at very high subsonic speeds, the problem of the existence of a local region of supersonic flow became important. People were concerned that shocks might develop to influence the behavior of the boundary layer, and thereby produce seriously increased drag. However, it was later realized that if reasonable and ingenious care were exercised to avoid strong shocks, the problem of highly increased resistance could be overcome.

In the meantime, the problem has stimulated a mathematical investigation of the possibility of obtaining smooth solutions of partial differential equations of mixed elliptic and hyperbolic types (parabolic on the line of transition). This is a very interesting mathematical development in itself, but its value for the practical problem is limited. The stringent mathematical conditions required for guaranteeing a smooth flow would exclude very *weak* shocks, whose occurrence does not really matter in the practical problem.

I cite this example to warn our colleagues against investing too much effort in problems posed by scientists and engineers before finding out the real nature and implications of the solution of the mathematical problem. In the example cited, the effort is not fruitless, because the mathematical problem is of interest in itself. In other cases, the loss could be almost total.

6. *Applied Mathematics as an Independent Discipline*

I now come to the central point of my talk, which is a plea for the establishment of a healthy community of applied mathematicians and for increased support of education and research in applied mathematics, especially the support of academic activities. We cannot expect the important role of applied mathematics to be fulfilled without a steady supply of applied mathematicians of very high caliber. It is not enough to have only pure mathematicians who would devote a small fraction of their efforts to applied mathematics. Neither is it sufficient to have only theoretical scientists who are unaware of mathematical methods used in other sciences. Applied mathematicians must be educated with a broad knowledge of applicable mathematics and an extensive exposure to the application of mathematics.

A common educational program for applied mathematicians is also needed to create a community spirit, without which applied mathematics cannot remain a healthy profession.

I think it is time that we should recognize applied mathematics as an independent discipline, fairly distinct from pure mathematics.

The difference in spirit between pure and applied mathematics is clear and often very great. In a very interesting and important article [13], Professor J. Schwartz warned mathematicians against the danger of single-mindedness, literal-mindedness, and simple-mindedness in dealing with scientific problems. The above examples also show clearly the difference between the approach of pure and applied mathematicians. Yet the need for the education of applied mathematicians in a comprehensive spirit described in this article has been generally neglected. It was recommended in [8, Recommendation 18]. Nevertheless, only a few schools have established such programs.

One might ask whether it is realistic to adopt this comprehensive approach. From my own examination of this matter, the answer is an unqualified "yes." I have recently read papers and books in ecology, mathematical theories of population, economics, combinatorial analysis, etc. I did not find the mathematics used to be *that* different one from the other. The mathematical methods used in biological sciences also do not differ from those in physical sciences (see [6]). Certainly, people use similar methods in numerical analysis and similar computing machines.

If the mathematical methods used in the various sciences were widely divergent, the establishment of a community spirit among applied mathematicians would of course be more difficult.

In this country, there had not been a tradition of applied mathematics. People now in this profession come from a variety of backgrounds. There is thus a definite need for practicing applied mathematicians to do something important for this community. This is to educate each other in the subject of his own special line of research in a well-motivated manner. This would create mutual understanding, establish mutual confidence, and build up the community spirit.

In the following sections, I present a discussion of the educational program and conclude with a number of recommendations. (In the oral presentation, this part was omitted.)

III. Education

7. *Education of Applied Mathematicians*

The ultimate aim of any liberal education program must include the following:

(1) the cultivation of an attitude, points of view, and value judgements;

(2) the acquirement of a vast body of coherent knowledge which deserves to be handed down from one generation to another, and which is best understood and special to this particular area of pursuit; and

(3) the development of certain talents, especially creative ability in the particular area in question.

From these, it is expected that the person will develop, through his personal experience in creative activity, a wisdom which enables him to act as an advisor to his co-workers and a teacher to the younger generation. In the latter capacity, he must be able to present a perspective of the total activity so that the younger generation will be able to decide how and where to devote their efforts, and to carry on their work with confidence. The lack of this perspective would often lead to wasteful efforts.

How can we arrange an educational program in applied mathematics to achieve the objectives described above?

First of all, the education must be started in the undergraduate years, during the formative period of the youth. If a person has formed an attitude to consider it far more valuable to prove an elegant theorem in algebra than to explain a phenomenon observed in the atmosphere or in the galaxies with the help of mathematical methods (or to consider the latter as "somebody else's business"), it would be very difficult to convert him to another set of beliefs. If a person has been impressed with a zeal to use his mathematical talent to elucidate a point in high energy physics, he might not be as enthusiastic in attacking a problem in biology or in economics.

The educational material (or the primary content of the knowledge of a student in applied mathematics) then must be based on an account of the typical impressive past accomplishments of mathematics selected from *all* the sciences. The method of case study must be used. At the same time, the students should also be made to understand that, while there would be a continuous opportunity for the applied mathematician in established areas (such as mechanics, or more broadly, physics) dramatic accomplishments of the future might lie in some other areas, where mathematics has not, as yet, been fully utilized. One must *avoid the danger of sterility by an overemphasis of traditional subject matter.*

It is the lack of survey courses of this broad nature that makes it difficult to educate future comprehensive applied mathematicians. A young student with an aspiration to become one usually finds himself in the almost impossible position of having to sift out for himself the mathematical aspects of the basic sciences from the courses that are available to him from the various scientific departments at a university. While this attempt itself is of great educational value, the total effort needed is overwhelming. The usual course of action for the student is then to attach himself to one of the traditional disciplines, either in mathematics or in one of the sciences. This specialization tends to defeat the major purpose of the education of comprehensive applied mathematicians.

The job of abstracting and synthesizing the total picture of the interplay between mathematics and the sciences must be done for these inspired young *students by the experienced scientists.* Indeed, this type of effort is the primary responsibility of the teachers in any profession. They must provide the introductory or survey courses to present the (undergraduate) students with a *total* picture of the *status quo* in the whole field so that future research efforts can be pursued with the greatest effectiveness.

Courses of another type, best taught by professional applied mathematicians, are those usually designated "methods of applied mathematics." These should be available at various levels. Besides these two types of "professional courses," the students must have a good foundation in basic pure mathematics, and some depth of education in at least one branch of science.

We may thus summarize a basic educational program in comprehensive applied mathematics[6] as follows:

(a) an education in the attitude of an applied mathematician,

(b) an education in the usual working ability (methods of applied mathematics),

(c) a survey in applied mathematics with emphasis on the total picture,

(d) a basic education in pure mathematics,

(e) An education in at least one branch of science in depth.

The subprograms (a), (b), and (c) should be normally taught by applied mathematicians; subprogram (d), by pure mathematicians; and subprogram (e), by professional scientists in various departments.

Graduate education should naturally be an extension and a continuation of the basic program discussed above; it must naturally include the development of the ability of the student to do research. The broad outlines of such abilities have been given in Section III. Here, as in all subjects, specialization is unavoidable, and the available lines of research at a given institution must depend on the faculty members available. It would, therefore, be useless to go into further details in this general discussions.

One should note the *central role of courses for subprogram* (c) (survey in applied mathematics), which should be approximately a two-year sequence in the junior and senior years. (It should be started in the

[6] In practice, this might mean three semester courses in pure mathematics, four semester courses in scientific subjects, and six semester courses in applied mathematics [(b) and (c)].

sophomore year, if possible.) Without these courses, we tend to educate only (i) pure mathematicians who might be occasionally interested in applications, or (ii) highly specialized applied mathematicians and theoretical scientists who would individually be very competent in a relatively narrow area of mathematics in his particular field of interest. These people do contribute to applied mathematics; but their presence alone is not sufficient for the healthy development of applied mathematics. The propagation of a comprehensive educational discipline of applied mathematics is essential to its success, since we must attract enterprising and brilliant young students in large numbers into the profession, and offer them a promising academic career. This last point is perhaps the most important point to the success of the development of applied mathematics. Applied mathematicians must be judged as such in terms of their total ability in promoting the interaction. They must also have the opportunity to educate their own future professionals.

Clearly, we do not advocate the education of applied mathematicians by first training them as pure mathematicians. Such a training has the danger of introducing a frame of mind which is disadvantageous to their creative activities. Those already educated in pure mathematics may wish to practice applied mathematics; but they should keep in mind the warnings of Professor Schwartz mentioned earlier in this paper.

8. *Some Practical Aspects of the Educational Program*

There are at least three important practical reasons for providing an undergraduate educational program in applied mathematics at a university.

(A) If only a graduate program is provided, one has to concentrate on education toward research ability in a certain specialized area; and there will not be time for education in breadth. This results in a splintering of activities in applied mathematics, making the total effect less effective. The usual sympton of difficulty is the small number of excellent students entering the graduate school to study applied mathematics.

(B) The maintenance of an undergraduate program would give the faculty in applied mathematics a common focus of interest and activity. The subject matter in an undergraduate program will evolve, but comparatively slowly (Newton mechanics still has its role, quantum mechanics will not be changed overnight). This permits a relatively stable joint effort on the part of the faculty. Joint effort creates stability.

In a graduate program, there is necessarily a diversification of interest among the faculty members. Furthermore, their interests often have to change quite rapidly in accordance with the current trends of scientific progress and the vitality of the subject matter. There cannot be real communication except among the specialists themselves. The complete lack of communication among colleagues is not healthy for any group.

(C) The courses in mathematical methods, and even the survey course, have tremendous educational value to many other students in pure mathematics, science, and engineering. Indeed, by a systematic offering of courses in mathematical methods, one may eliminate the need to provide special "service courses," which have often been the source of great discontent at a university.

A faculty member is usually required to fulfill a certain minimum teaching obligation. With the undergraduate program, the teaching obligations of an applied mathematician would be essentially to offer courses for the education of future applied mathematicians. The usefulness of these courses for the other purposes mentioned above is a natural beneficial side effect. If the applied mathematicians must teach mostly "service courses," they cannot help the feeling of being "second-class citizens" in the academic community.

IV. Recommendations

9. *Support of Academic Activities in Applied Mathematics*

To prevent mathematics from drifting further and further away from science and technology, there is a need to develop academic applied mathematics: There is a need to have highly qualified scientists specifically devoted to such activities. I wish to make the following recommendations toward accomplishing this goal.

Recommendation 1. *The university administration and the scientific community, including the pure mathematicians, should be made keenly aware of the above-mentioned need.*

Recommendation 2. *Programs in academic applied mathematics should* *be offered in the universities, at both the undergraduate and graduate levels, aiming at the highest intellectual caliber.*[7]

At first, because of the shortage of applied mathematicians, these programs could be offered by the cooperative efforts of scientists and engineers together with pure mathematicians who have an interest in applications. But it is clear that the service teaching mentality would be detrimental to the morale. Persons of highest scientific caliber, whose primary efforts are elsewhere and who must work in a highly competitive atmosphere, rarely can be depended upon to offer their time and energy to "other people's business" for an extended period of time. We therefore need professional applied mathematicians to educate future professional applied mathematicians, whose ability and contributions are to be judged in terms of the creative activities described above and in terms of promoting the interaction between mathematics and the sciences.

We therefore arrive at the following most important recommendation.

Recommendation 3. *The universities should create faculty positions specifically in Applied Mathematics and appoint to these positions only people of the highest caliber.*

Their research activities should be well respected in the scientific community. Judgement of their work should be based on their contributions to the effective interaction between mathematics and science.

Since the National Science Foundation is awarding research grants to support all branches of pure science, the academic aspect of applied mathematics (AAM) must not be an exception. Otherwise, due to the pressure of the needs of mission-oriented support,[8] the academic nature of such work, with all its long-range benefits, would be jeopardized. We therefore offer

Recommendation 4. *The National Science Foundation should continue to expand its explicit support to academic activities in applied mathematics,* as such (not under the guise of research in some subject matter). Those activities, whose usefulness is easily identifiable, may continue to be supported by mission-oriented agencies. A plurality of pattern of support should be maintained.

[7] A secondary beneficial effect of such programs is the elimination of the need to offer service courses of low caliber to students in engineering and science.

[8] It must be said that, at present, the mission-oriented agencies have been very generous in their support of basic research whose usefulness may not be directly apparent. But this is a policy susceptible to pressure from the U.S. Congress.

V. Appendix

Toward New Horizons with Great Traditions

While the newly developed and developing subjects in applied mathematics would naturally attract greater attention, the role of applied mathematics developed in the classical spirit should also be fully appreciated in future research, and especially as a vehicle for education. The usual danger of emphasizing all kinds of traditional effects whether in science or in other fields is to fall into a pit of sterility. It must be avoided by accepting new challenges with a flexible attitude. With this in mind, one can indeed make the best of a wealth of knowledge made available to us through efforts of the great minds of past generations.

(a) *Education*

While excitement is often created by new developments in any scientific subject, traditional subject matter, sifted through many generations of critical thinking and brought up to its modern form, can often serve as a core of knowledge on which future developments can be based. It is indeed these crystalized thoughts, ideas, and reasoning, which must be taught in the undergraduate years as a source of stimulation and as a basis for future research efforts. Indeed, a common background of undergraduate education would serve as a basis of establishing rapport among applied mathematicians who might later specialize in different lines of research.

Needless to say, the pattern of education and detailed choice of subject matter must evolve in the course of time as our perspective over the total picture of applied mathematics changes in view of new developments.

(b) *Research*

The nature of research efforts in the "traditional" line actually evolves rapidly and substantially in the course of time. These research efforts also bring out clearly unsolved classical problems whose implications are obviously important but yet not fully understood. As an example in the latter category, the study of "turbulence" in the context of hydrodynamics is still an unfinished task. It is now clear that the existing studies and the still unresolved problems both relate to the broader problem of "nonlinear random processes." In the case of incompressible viscous fluids, the physical processes are very well understood from a general point of view, and many detailed comparisons of theoretical and observed results have become possible; but the fundamental mathematical theory remains unformulated.

Stability and instability, wave motion, linear versus nonlinear processes, reversible versus irreversible processes, entropy, and one can name some more, are certainly classical topics whose implications are general. The concept of entropy has already found its new application in information theory. It is quite possible that the concept of wave motion and the description of the process of shock wave formation might turn out to be useful in the mathematical description of social processes. On the other hand, one can easily trace the change of direction and find new vitality in an old subject like fluid mechanics. Around the turn of the century, aerodynamics, as related to aeronautical engineering, was one of the great challenges of the times. It remained strong as a subject for research of applied mathematicians for a long time. Nowadays, aerodynamics is so well understood that it is no longer the major concern of the applied mathematician. In its place, other aspects of fluid mechanics, those related to geophysical and astrophysical problems, are attracting greater attention. Certain concepts developed primarily in aerodynamics, such as the boundary layer, however, remain important even in these newer studies.

Continuum mechanics is of course but one aspect of classical applied mathematics, although it does have a special position, at least for the following three reasons: (i) the physical concepts are simple, (ii) the mathematical problems involved are interesting and of wide applicability, and (iii) a variety of natural phenomena fall within its scope. Particle dynamics also occupied and continues to occupy an important position in applied mathematics. There is currently a great deal of activity based on the theory of dynamical systems (integrals of motion, adiabatic invariants, etc.) in the construction of models of galaxies of stars. Even the whole subject of mechanics and statistical mechanics is but one part of "classical" mathematical physics. There will no doubt remain many problems to be explored in the whole broad area of such activities for years to come. The solid foundation of traditional efforts will always help us to reach new horizons if we adopt the progressive point of view.

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